New tools for surface analysis

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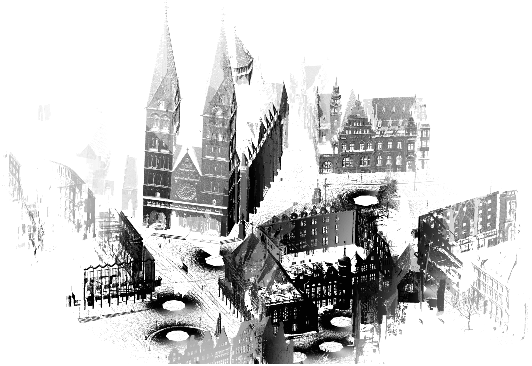
Sampled surfaces

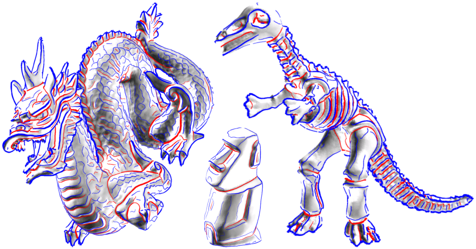

Musée de Lyon Fourvière, LIRIS, projet PAPS

Sampled surfaces

Local Shape Analysis

- Surface normals
- Surface Curvatures
- Curvature lines



[Ohtake et al. 2004]

Estimation

Need to estimate differential quantities on sampled surfaces.
⇒ Can be irregularly sampled, noisy, missing data.

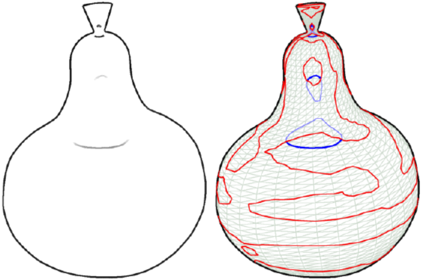
Curvature Estimation

On point sets:

- Osculating Jets [Cazals 03], Wavejets [Béarzi 2018]
- Voronoi Curvature Measure [Mérigot 10]
- Curvature tensor estimation [Kalogerakis 07,09]

On meshes

- Curvature and Curvature derivatives estimation [Rusinkiewicz03]
- Normal Cycles [Morvan, Cohen-Steiner 03]
- Laplace Beltrami discretization [Meyer02, Wardetzky07, Vallet08]



Per point/vertex computation

Tangent Vector Fields

Goal

Compute a smooth tangent vector field with user-prescribed constraints optimizing some regularity criterion.

- N-symmetry direction fields [Ray 08]
- Equivalent to a Riemannian metric design problem [Lai 10]
- Smoothness constraints [Crane10,Knoppel13], symmetry constraints [Panozzo14]

More global methods: permit to constrain directions from a global point of view.

Higher order Information?

- Curvature derivatives: helps finding suggestive contours [Rusinkiewicz 03]

In this talk

Can we define principal directions of higher order, and would they reveal something on the surface?

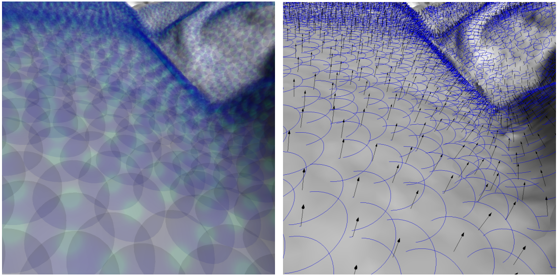
Assumptions

Underlying surface S:

- S can locally be expressed as a height field over a planar parameterization in neighborhoods of *fixed* radius r
- S is smooth, $\mathcal{C}^{\infty}$

Discretization

- *Sampling condition:* r-neighborhood of a seed containing enough points.
- *Noise level:* Noise magnitude strictly below radius r.



Local surface representation

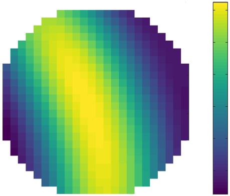
Height-fields

- Height-field over a plane:

$$\boldsymbol{p}(x, y, h = f(x, y))$$

- Taylor expansion at $(0, 0)$

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{1}{(k-i)!i!} \frac{\partial^k f}{\partial x^i \partial y^{k-i}}(0, 0) x^i y^{k-i}$$



A small detour by symmetric tensors

Def. symmetric tensor

A m-dimensional symmetric tensor T of order k is a m-dimensional array such that given index $I = (i_j)_{j \in [\![0,m]\!]}$, for any permutation p on I, $T_I = T_{p(I)}$

- $m = 2$ let $\boldsymbol{v} = (x, y)$, $T = (T_x, T_y)$ symmetric tensor of order k, then $T\boldsymbol{v} = xT_x + yT_y$.
- $T\boldsymbol{v}$ is a symmetric tensor of order $k - 1$
- $T\boldsymbol{v}^j$ is the result of contracting T by $\boldsymbol{v}$ j times.

E-eigenvalues of symmetric tensors

Eigenvalues [Qi 2005,2006,2007]

Given T a symmetric tensor of order k, if there exists $\lambda \in \mathbb{C}$ and a vector $\boldsymbol{v} \in \mathbb{R}^2$ such that:

$$\begin{cases} T\boldsymbol{v}^{k-1} & = & \lambda \boldsymbol{v} \\ \boldsymbol{v}^T\boldsymbol{v} & = & 1 \end{cases} \tag{1}$$

Then λ is called an E-eigenvalue of T and $\boldsymbol{v}$ is called an E-eigenvector of T. The set of λ satisfying (1) are the roots of a polynomial called the E-characteristic polynomial.

Disclaimer

Nomenclatura: Supermatrix [Qi] or Tensor.

Arbitrary order differential tensor

Differential tensor

f defined on $\mathbb{R}^2$ with values in $\mathbb{R}$. T_k is a symmetric tensor of order k, where coefficients are as follows: let $x_0 = x, x_1 = y$,

$$(T_k)_{(i_0,\dots,i_k)} = \frac{\partial^k f}{\partial x_{i_0} \dots \partial x_{i_k}}(0,0) \tag{2}$$

$$T_k \boldsymbol{v}^k = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f}{\partial^i x \partial^{k-i} y}(0,0) x^i y^{k-i} \tag{3}$$

Writing f with T_k

$$f(\boldsymbol{v}) = \sum_{k=0}^{\infty} \frac{1}{k!} T_k \boldsymbol{v}^k + o(\|\boldsymbol{v}\|^K)$$

Tensor differentiation

Expansion

Differentiating a symmetric tensor of order k, $T(v)$ wrt a vector v yields a symmetric tensor of order $k+1$

Lemma

Let T be a symmetric tensor. Let $\boldsymbol{v} = (x, y)^T \in \mathbb{R}^2$ be a vector.

$$\frac{\partial T \boldsymbol{v}^k}{\partial \boldsymbol{v}} = k T \boldsymbol{v}^{k-1} \tag{4}$$

Eigenvectors

Theorem

Given $\boldsymbol{v} = (x, y)$, T_k a real symmetric tensor of order $k > 1$ representing the derivatives of order k of a smooth function f in $\mathcal{C}^k$, the set of vectors $\boldsymbol{v} = (x, y) = (r\cos\theta, r\sin\theta)$ such that $\frac{\partial}{\partial\theta} T_k \boldsymbol{v}^k = 0$ and $\|\boldsymbol{v}\| = 1$ are E-eigenvectors of T_k:

$$\begin{cases} T_k \boldsymbol{v}^{k-1} = \boldsymbol{v}\, T_k \boldsymbol{v}^k \\ \|\boldsymbol{v}\| = 1 \end{cases} \tag{5}$$

Sketch of the proof

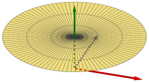
- Show that $\frac{\partial}{\partial r} T_k \boldsymbol{v}^k = \frac{k}{r} T_k \boldsymbol{v}$
- Show that $T_k \boldsymbol{v}^{k-1} = \frac{T_k \boldsymbol{v}^k}{\|v\|^2} \boldsymbol{v}$
- Since $\|v\| = 1$ and by setting $\lambda = T_k \boldsymbol{v}^k$, we get $T_k \boldsymbol{v}^{k-1} = \lambda \boldsymbol{v}$.

Expressing T in the *Wavejets* basis

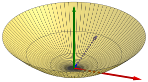
- Switching to polar coordinates $\boldsymbol{v} = (x, y) = (r\cos\theta, r\sin\theta)$
- Wavejets Basis definition: [Béarzi Digne Chaine 18]

$$f(r,\theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} B_{k,n}(r,\theta) = \sum_{k=0}^{\infty} \sum_{n=-k}^{k} \phi_{k,n} r^k e^{in\theta}$$

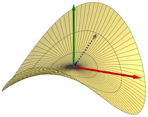
Wavejets Basis [Béarzi et al. 2018]



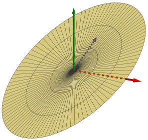
$B_{0,0}$



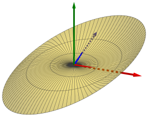
$B_{2,0}$



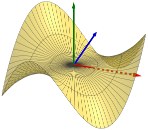
$B_{2,2} + B_{2,-2}$



$B_{1,1} + B_{1,-1}$



$B_{3,1} + B_{3,-1}$



$B_{3,3} + B_{3,-3}$

Consequence

Corollary

Given $\boldsymbol{v} = (x, y)$, the directions of the E-eigenvectors of a tensor T_k of order k can be retrieved out of the Wavejet decomposition of $T_k \boldsymbol{v}^k$ by looking at the zeros of:

$$\frac{\partial}{\partial \theta} \sum_{n=-k}^{k} \phi_{k,n} e^{\boldsymbol{i} n\theta} = \sum_{n=-k}^{k} \boldsymbol{i} n \phi_{k,n} e^{\boldsymbol{i} n\theta} \tag{6}$$

Principal directions

For any order k we can extract eigenvectors of the k^{th} order symmetric tensor corresponding to the k^{th} order differential tensor

Maximum and Minimum Arbitrary Order Principal Directions

Definition

Maximum principal directions (resp. *minimum principal directions*) are defined as local maxima (resp. local minima) of

$$g_k(\theta) = \sum_{n=-k}^{k} \phi_{k,n} e^{\boldsymbol{i} n\theta} = \frac{T_k \mathbf{v}^k}{k! r^k}$$

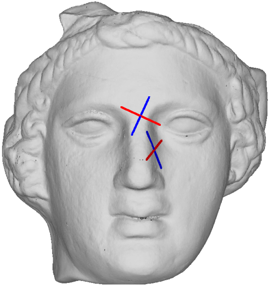
with $\mathbf{v} = (r\cos\theta, r\sin\theta)$.
The corresponding eigenvalues are given by: $\lambda = k! g_k(\theta_0)$ with θ_0 the angle corresponding to a maximum or minimum principal direction.

Order 2 Principal directions (aka Curvature principal directions)

- Principal curvatures:

$$\kappa_1 = 2\left(\phi_{2,0} + \phi_{2,2} + \phi_{2,-2}\right) \text{ and } \kappa_2 = 2\left(\phi_{2,0} - \phi_{2,2} - \phi_{2,-2}\right) \tag{7}$$

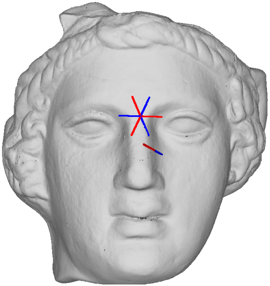
- $\sum_{\substack{-2\leq n\leq 2\\ \text{n even}}} \phi_{2,n} e^{\boldsymbol{i}n\theta} + \phi_{2,-n} e^{-\boldsymbol{i}n\theta}$ has 2 maxima aligned with the principal directions



Higher order principal directions

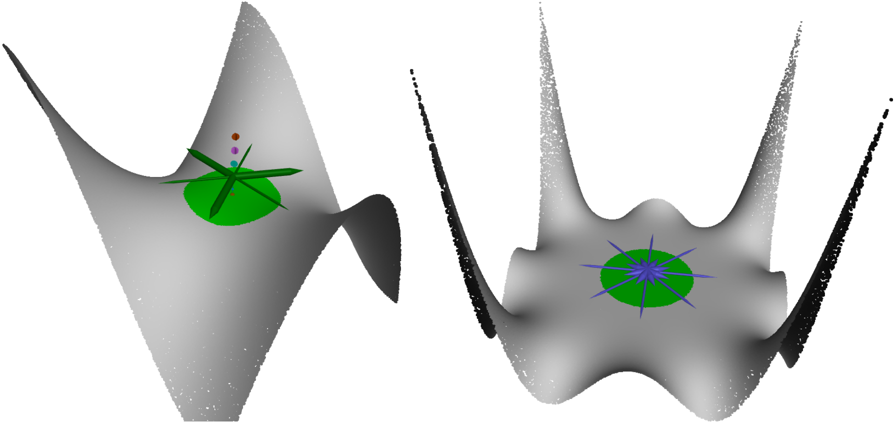
Order 3

- $\sum_{\substack{-n\leq 3 \\ \text{n odd}}} \phi_{3,n} e^{\boldsymbol{i}n\theta} + \phi_{3,-n} e^{-\boldsymbol{i}n\theta}$ has at most 3 maxima (either 1 or 3)



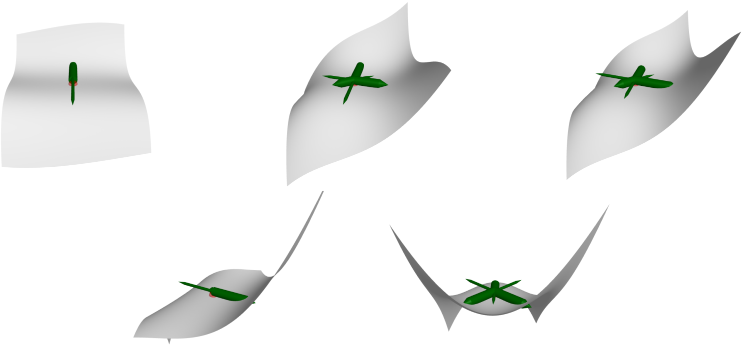
Order 3 maxima directions

Synthetic Examples



Two synthetic surfaces with relevant principal directions of order 3 and order 8.
Other orders vanish and exhibit no principal directions.

Synthetic Examples



Order 3 principal directions on a synthetic surface controlled by its Wavejets coefficients.

Properties of order k directions

- If k is even: if θ_0 corresponds to a maximum principal direction, $\theta_0 + \pi$ also corresponds to a maximum principal direction.
- If k is odd: if θ_0 corresponds to a maximum principal direction, $\theta_0 + \pi$ corresponds to a minimum principal direction.
- At most $2k$ principal directions of order k (roots of a real polynomial of order $2k$)
- Regularity: Order k principal directions are regular iff $\phi_{k,n} = 0$ fo $n \neq \pm k$.

Practical computation: Truncating the Taylor Expansion

Osculating Jets [Cazals03]

- Surface parameterized w.r.t. $\mathcal{P}(p)$ Not necessarily equal to $\mathcal{T}(p)$ (tangent plane)

Truncated Taylor expansion

$\mathcal{S}$ surface locally homeomorphic to a disk in a small neighborhood around a point $\boldsymbol{p}$, expressed as $f(x,y)$ over a plane $\mathcal{P}(\boldsymbol{p})$ passing through $\boldsymbol{p}$. The neighborhood of $\boldsymbol{p}$ can be expressed as a *truncated* Taylor Expansion at order K:

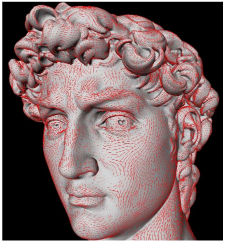
$$f(x,y) = \sum_{k=0}^{\infty} \sum_{j=0}^{K} \frac{f_{x^{k-j}y^j}(0,0)}{(k-j)!j!} x^{k-j} y^j \tag{8}$$

where $f_{x^{k-j}y^j} = \frac{\partial^k f}{\partial x^{k-j} \partial y^j}$.

Practical computation: Truncation order

Accuracy theorem [Cazals03]

Given a Taylor expansion of order K in a neighborhood of radius r, the precision of all k order derivatives is $o(r^{K-k})$.



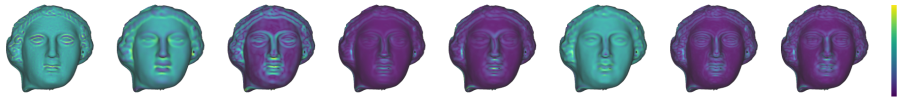
[Cazals 2003]

- In practice: Computation of the coefficients at each vertex or point by linear system solve.

Practical computation

Wavejets

The Wavejets expansion can be truncated similarly to the Osculating Jets expansion.



$\phi_{0,0}$ $\phi_{2,0}$ $|\phi_{2,2}|$ $|\phi_{3,1}|$ $|\phi_{3,3}|$ $\phi_{4,0}$ $|\phi_{4,2}|$ $|\phi_{4,4}|$

- $(r_\ell, \theta_\ell, h_\ell)_{\ell \in [\![1,N]\!]}$: local coordinates around $\boldsymbol{p}(0,0)$

$$\underbrace{\begin{pmatrix} B_{0,0}(r_1,\theta_1) & B_{1,-1}(r_1,\theta_1) & B_{1,1}(r_1,\theta_1) & \cdots & B_{K,K}(r_1,\theta_1) \\ B_{0,0}(r_2,\theta_2) & B_{1,-1}(r_2,\theta_2) & B_{1,1}(r_2,\theta_2) & \cdots & B_{K,K}(r_2,\theta_2) \\ \vdots & & & & \vdots \\ B_{0,0}(r_N,\theta_N) & B_{1,-1}(r_N,\theta_N) & B_{1,1}(r_N,\theta_N) & \cdots & B_{K,K}(r_N,\theta_N) \end{pmatrix}}_{M} \times \underbrace{\begin{pmatrix} \phi_{0,0} \\ \phi_{1,-1} \\ \phi_{1,1} \\ \vdots \\ \phi_{K,K} \end{pmatrix}}_{\Phi} = \underbrace{\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}}_{H}$$

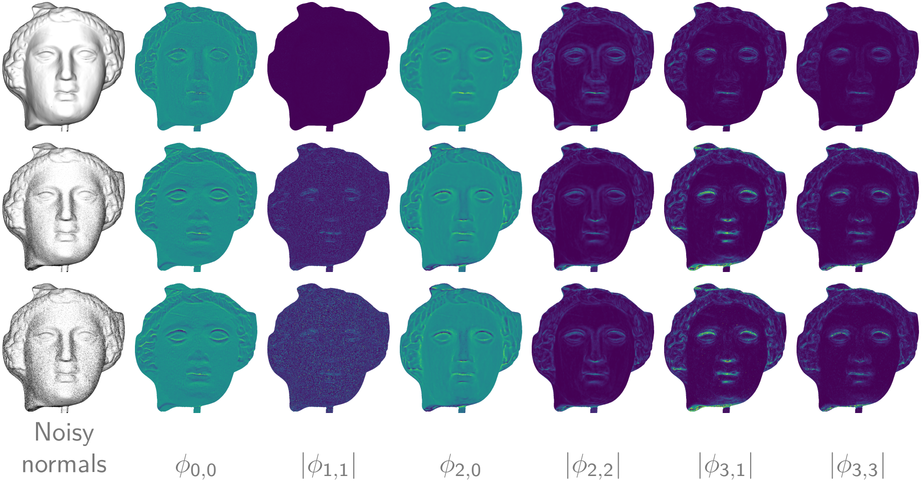
- Solve using QR decomposition

$$\underset{\Phi}{\operatorname{argmin}} \|M\Phi - H\|^2$$

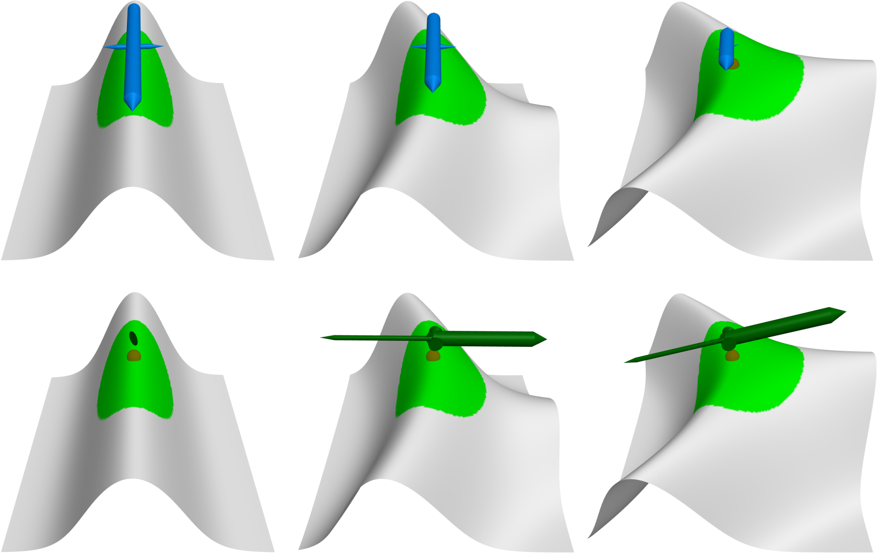
Properties

- Adding a weight depending on the distance of the neighbor to p
- If the weight is smooth and radially decreasing:
  - ℓ^2 regression yields smooth coefficients [Levin15]....
  - ℓ^1 no such guarantee.

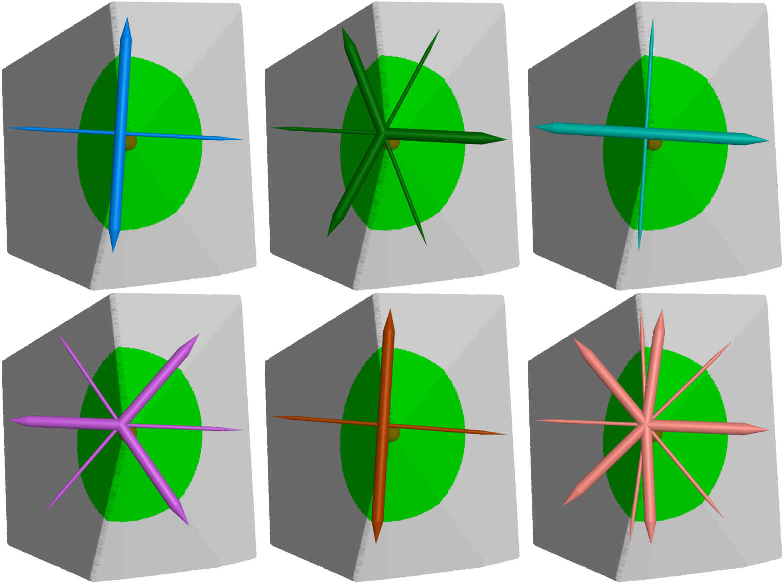
Results



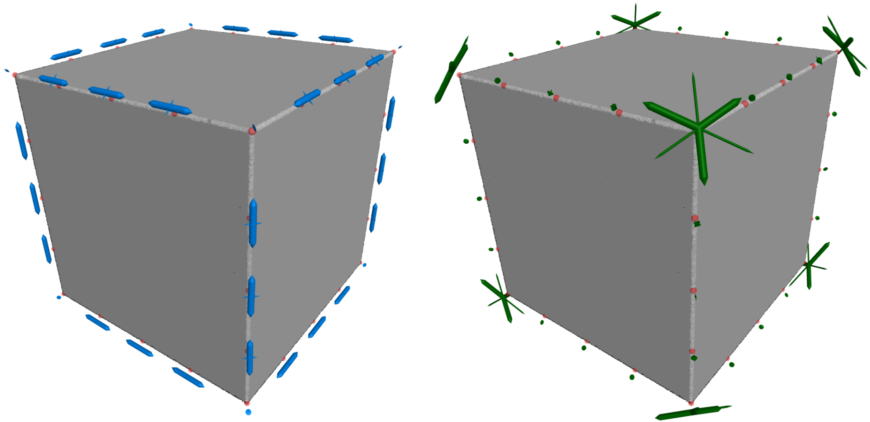
Experiments



Order 2 (top) and 3 (bottom) principal directions on a surface evolving from a ridge (left) to a smooth T-junction (right).

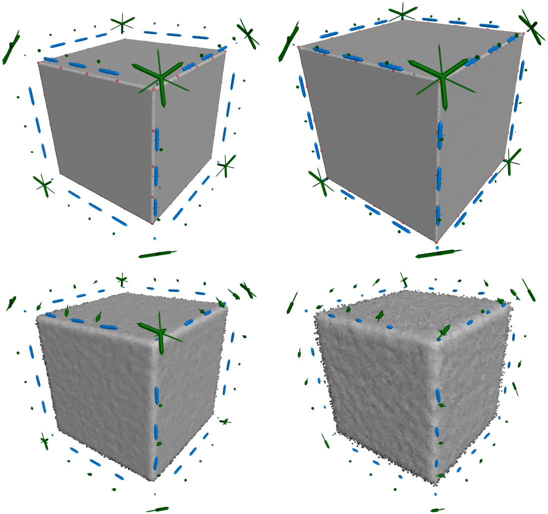


Orders 2 to 7

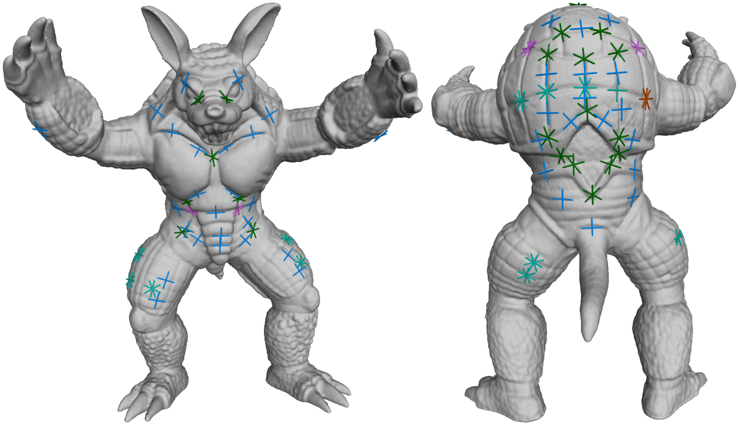


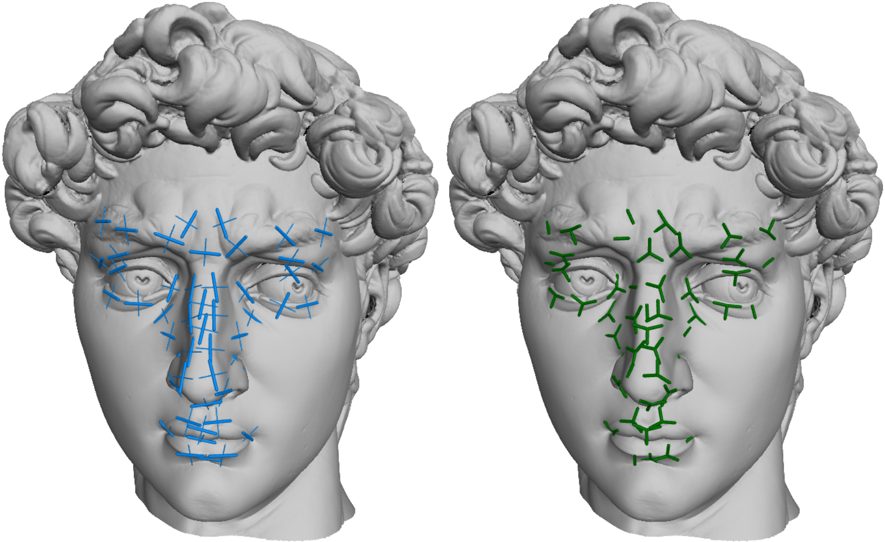
Orders 2 and 3

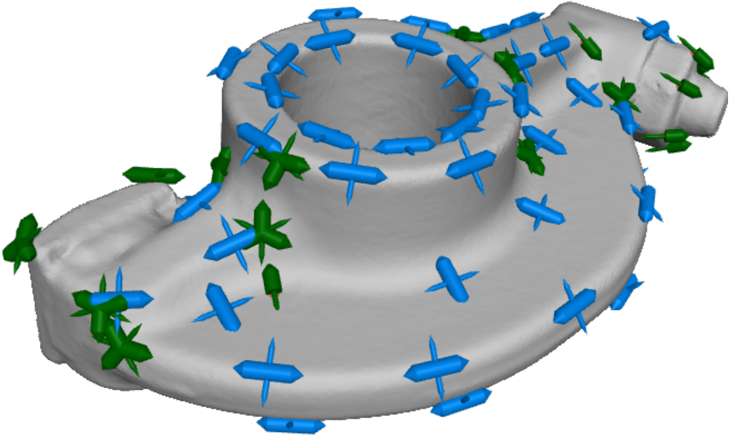
With noise

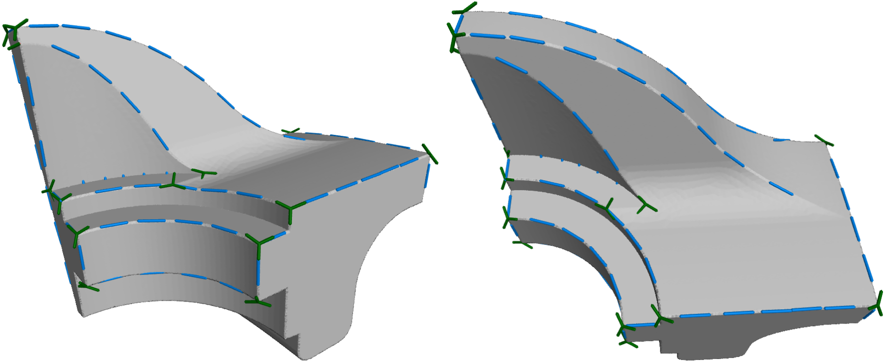


Principal directions of order 2 and 3 computed on a cube with added Gaussian noise on the positions. Top: Noiseless, $\sigma = 0.01\%$; Bottom: $\sigma = 0.05\%$ and $\sigma = 0.1\%$

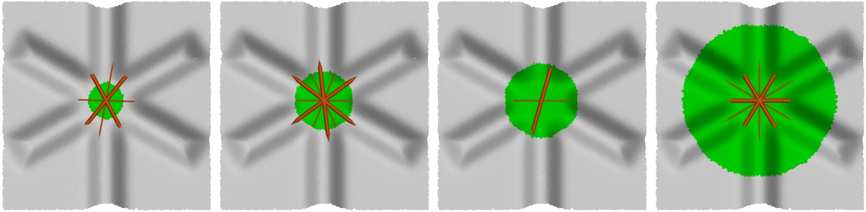








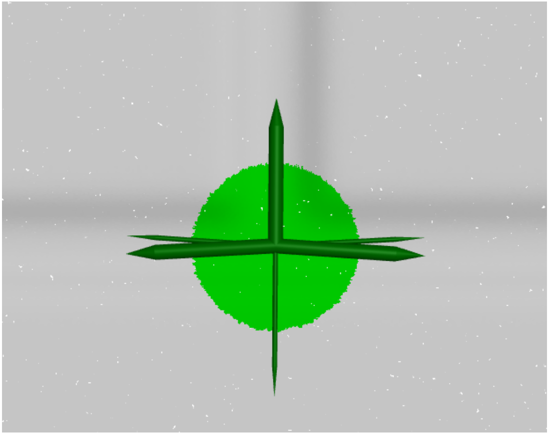
Dependency on the radius



Estimation with $r = 50, 80, 100, 200$.

Limitations

- Parameters: radius r, truncation order K
- Distribution of the principal directions of a given order are not arbitrary!



Disclaimer

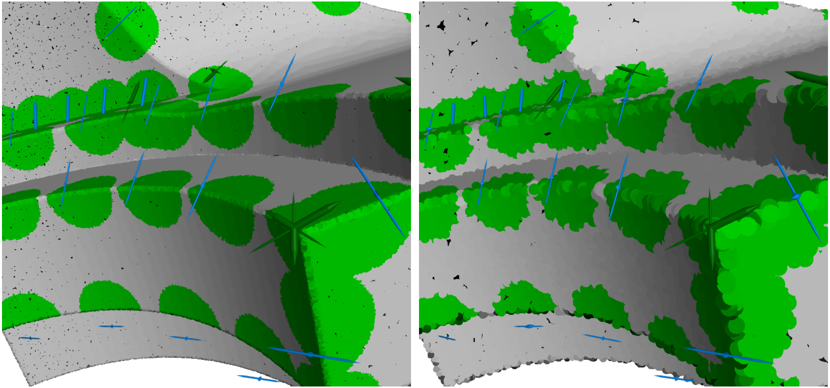
Likely crash due to the demo effect.

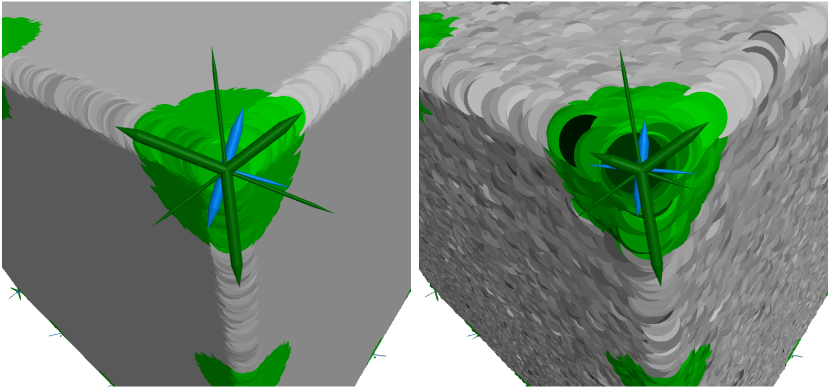
Application: Line tracking

- Follow a line and increase order when lowest order becomes meaningless

Application: Shape registration

- When a higher order with a least three principal directions becomes meaningful, the point can be well localized
- Rigid transform with a single point and its direction field.





Conclusion

- Extension of principal directions to *any* differential order
- Easy computation in the Wavejets basis
- Application to geometry processing, and more to explore

References:

- *Arbitrary order principal directions and how to compute them*, J. Digne, S. Valette, R. Chaine, Y. Béarzi, Preprint Nov. 2021 ArXiv Preprint 2111.05800
- *Wavejets: A Local Frequency Framework for Shape Details Amplification*, Y. Béarzi, J. Digne, R. Chaine 2018.