Binary Morphological Neural Networks

Theodore Aouad, Hugues Talbot

Center for Visual Computing - OPIS Inria - CentraleSupelec - University of Paris-Saclay

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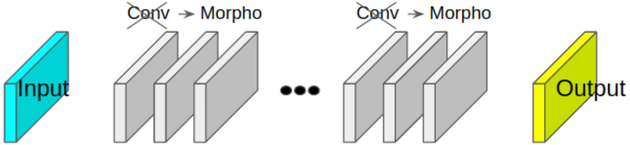
From convolution to morphological operators

- Convolution filters have been very successful in many vision problems.
- Convolutional Neural Networks (CNNs) learn the best filters for a given task.
- However, they are still cases where mathematical morphology makes more sense than convolution.
- Replacing "convolution" of CNNs by basic morphological operators (dilation and erosion) could be useful.
- How can we learn the structuring elements and the right sequence of operations for a given problem?

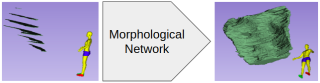


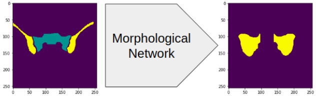
Related Work

- Learning morphological operators is not new. (Wilson [1993], Nakashizuka et al. [2010], Barrera et al. [1997])
- Recent hype on deep learning has motivated new techniques.
- Some researchers use the max-plus and min-plus definition of the dilation and erosion to perform grey-morphology on grey-scale images. (Mondal et al. [2019, 2020], Franchi et al. [2020])
- Others replace the max operation by a softmax. (Masci et al. [2013], Kirszenberg et al. [2021], Shen et al. [2019])
- Others try to learn a binary SE for grey-scale morphology (Nogueira et al. [2021])

Motivation

- Related research has primary worked on grey-scale images, with either grey-scale or binary structuring elements.
- Our aim is to perform shape analysis. For example, we want to be able to detect useful ROIs given input regions, or to infer the 3D shape using only a few slices.
- To do so, we want a system that can take work with fully binary inputs and binary structuring elements.





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Morphological operator from convolution

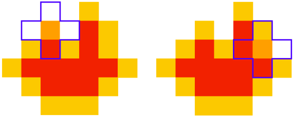
We rewrite the classical Minkowski addition and its dual operation with convolution.

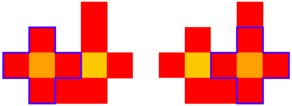
Proposition (Convolution for morphological operators)

Let $S \subset \mathbb{Z}^d$ be a binary structuring element and $X \subset \mathbb{Z}^d$ be a binary image.

$$X \oplus S = \left(\mathbb{1}_X \circledast \mathbb{1}_S \geq 1\right) = \left\{ j \in \mathbb{Z}^d \middle| (\mathbb{1}_X \circledast \mathbb{1}_S)(j) \geq 1 \right\} \tag{1}$$

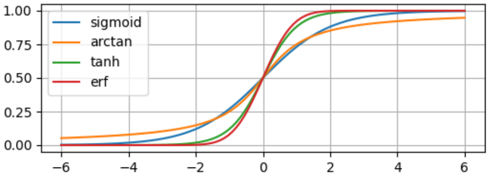
$$X \ominus S = \left(\mathbb{1}_X \circledast \mathbb{1}_S = card(S)\right) = \left\{ j \in \mathbb{Z}^d \middle| (\mathbb{1}_X \circledast \mathbb{1}_S)(j) = card(S) \right\} \tag{2}$$





Thresholding Weights

- Find $S \subset \mathbb{Z}^d$ using a weights matrix $W \in \{0,1\}^\Omega$, with Ω the support of the weights matrix (typically $\Omega = [-n, n]^d \cap \mathbb{Z}^d$, with $S \subset \Omega$).
- Relax W to be smooth and use a smooth thresholding $\xi : \mathbb{R} \mapsto]0,1[$ to ensure $\xi(W) \in [0,1]^\Omega$
  1. ξ must be increasing
  2. $\xi(0) = 0.5$
  3. $\lim(\xi(x))_{x \to -\infty} = 0$ and $\lim(\xi(x))_{x \to +\infty} = 1$



Binary Sructuring Element (BiSE) neuron

Definition (BiSE neuron)

Let $W \in \mathbb{R}^{\Omega}$ be a weight matrix, $b \in \mathbb{R}$ a bias, ξ a smooth thresholding function and $p \in \mathbb{R}_{+}^{*}$ a scaling number. We define a **BiSE neuron** as follow:

$$\epsilon_{W,b,p} : x \in [0,1]^{\mathbb{Z}^d} \mapsto \xi(p(x \circledast \xi(W) - b)) \in [0,1]^{\mathbb{Z}^d} \quad (3)$$

- The BiSE neuron is able to learn both an erosion and a dilation, as well as the associated structuring element.
- The weights W learn the structuring element.
- The bias b determines the operation, either dilation or erosion.
- The scaling number p determines how close to binary the output is.

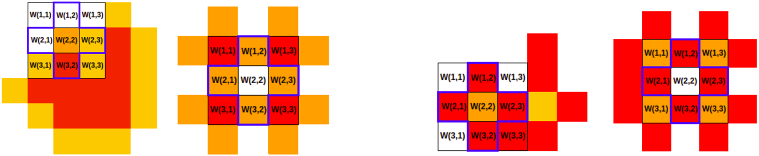
Equivalence BiSE / Dilation and Erosion - Binary input

Proposition

We assume the weights are thresholded: $W \in [0,1]^{\Omega}$. *Given a binary input,* $\epsilon_{W,b,+\infty}$ *is*

$$\text{a dilation by } S \text{ if and only if } \sum_{i \notin S} w_i \leq b < \min_{i \in S} w_i \quad (4)$$

$$\text{an erosion by } S \text{ if and only if } \max_{j \in S} \left(\sum_{i \neq j} w_i \right) \leq b < \sum_{i \in S} W_i \quad (5)$$



BiSE Output

Proposition (BiSE Output)

Let $u_1,\ u_2$ the bounds of the bias for dilation or erosion. (Ex for dilation, $u_1 := \sum_{i \notin S} w_i$, $u_2 := \min_{i \in S} w_i$). We assume that $u_1 \leq b < u_2$. Then we have:

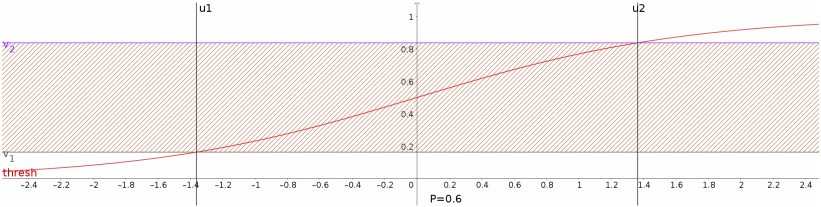
$$I \circledast W \notin \ ]u_1, u_2[\tag{6}$$

$$\epsilon_{W,b,p}(I) \notin \ ]\xi(p(u_1 - b)), \xi(p(u_2 - b))[\tag{7}$$

BiSE Output

Best case scenario, we have $b = \frac{v_1(p)+v_2(p)}{2}$. We show the possible output values depending on p.

$x \notin]u_1, u_2[\mapsto \xi(p * x) \notin]v_1(p), v_2(p)[$, with $x = I \circledast W(j) - \frac{u_1+u_2}{2}$:



Almost binary input

When a BiSE neuron is properly learned, the output will be either close to 1 or close to 0. Does it make sense to stack BiSE neurons if the outputs are not binary?

Definition (Almost Binary)

We say an image $I \in [0, 1]^{\mathbb{Z}^d}$ is **almost binary** if there exists $v_1 < v_2 \in [0, 1]$ such that $I(\mathbb{Z}^d) \notin]v_1, v_2[$.

We can extend the previously seen equivalence to almost binary inputs.

Equivalence BiSE / Dilation and Erosion - Almost Binary input

Proposition (Dilation Equivalence)

We assume the weights are thresholded: $W \in [0,1]^\Omega$. Given an almost binary input, $\epsilon_{W,b,+\infty}$ is a dilation by S if and only if

$$\sum_{i \notin S} w_i + v_1 \sum_{i \in S} w_i \leq b < v_2 \min_{i \in S} w_i \tag{8}$$

Proposition (Erosion Equivalence)

We assume the weights are thresholded: $W \in [0,1]^\Omega$. Given an almost binary input, $\epsilon_{W,b,+\infty}$ is an erosion by S if and only if

$$\max_{j \in S} \left(\sum_{i \neq j} (w_i) + v_1 w_j \right) \leq b < v_2 \sum_{i \in S} W_i \tag{9}$$

Binary Morphological Neural Network (BiMoNN)

Given an almost binary input, a properly trained BiSE neuron will always output an almost binary input. This allows the stacking of BiSE neurons sequentially.

Definition (Binary Morphological Neural Networks (BiMoNN))

We call a **Binary Morphological Neural Networks (BiMoNN)** a composition of multiple BiSE neurons. Let $K \in \mathbb{N}^*$, let $W = (W_1, ..., W_K) \in \mathbb{R}^{\Omega_1} \times ... \times \mathbb{R}^{\Omega_K}$ the set of weights for each BiSe neuron, $b = (b_1, ..., b_K) \in \mathbb{R}^K$ the set of biases and $p = (p_1, ..., p_K) \in \mathbb{R}^K$ the set of scaling numbers. We denote the BiMoNN as:

$$\phi_{W,b,p} = \epsilon_{W_K, b_K, p_K} \circ ... \circ \epsilon_{W_1, b_1, p_1} \quad (10)$$

Deep Learning Optimization

- Given a set of N couples of inputs-targets $\{(X_i, Y_i) \mid i \in \{1, ..., N\}\}$, given a loss function $\mathcal{L} : \mathbb{R}^2 \mapsto R$, we minimize:

$$\min_{W,b,p} \frac{1}{N} \sum_{i=1}^{N} \mathcal{L}(\phi_{W,b,p}(X_i), Y_i) \qquad (11)$$

- We optimize the loss using derivatives of stochastic gradient descent (batch-SGD, ADAM, ...).
- The BiMoNN is totally differentiable, and its convolutional structure make it optimized to compute the gradient for each parameter using back propagation.

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Training Data Generation

We train on generated binary data. The data is generated as follow:

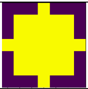
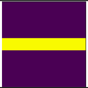
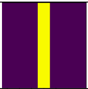
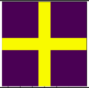
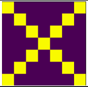
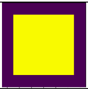
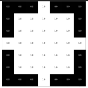
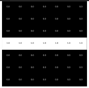
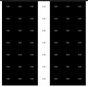
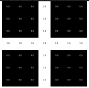
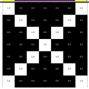
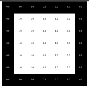
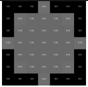
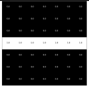
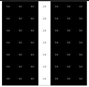
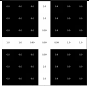
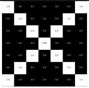
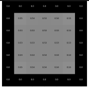
1. We generate $N = 30$ shapes
2. A shape is either a disk or a rotated rectangle box
3. We add random Bernoulli noise
4. We apply complementation with probability 0.5
5. We set the borders at 0 (depending on the kernel size Ω)

Training Regimen

- We tested on dilation, erosion, opening, closing.
- We tested 6 structuring elements of size 7x7.
- We generate images X_i and true targets Y_i with the true operation.
- For dilation and erosion, we train on $N = 200k$ images. For opening and closing, we train on $N = 1M$ images.
- The loss function $\mathcal{L}$ we use is the binary cross-entropy:

$$\mathcal{L}(\hat{y}, y^*) = y^* \log(\hat{y}) + (1 - y^*) \log(1 - \hat{y}) \tag{12}$$

Results - Final Weights on Dilation / Erosion

Operation	Disk	Hstick	Vstick	Scross	Dcross	Square
Target						
Dilation $\oplus$	 OK	 OK	 OK	 OK	 OK	 OK
Erosion $\ominus$	 OK	 OK	 OK	 OK	 OK	 OK

Results - Final Weights on Opening / Closing

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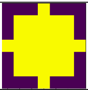
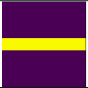
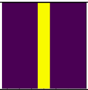
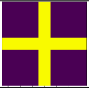
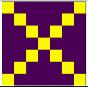
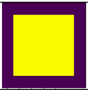
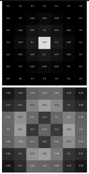
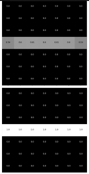
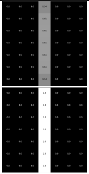
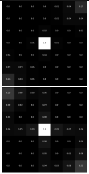
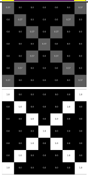
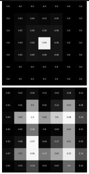
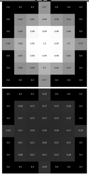
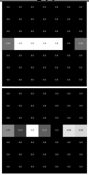
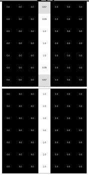
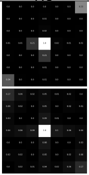
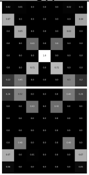
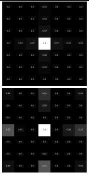
Target						
Opening ∘	 KO	 OK	 OK	 KO	 OK	 KO
Closing •	 OK	 KO	 OK	 KO	 KO	 KO

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Conclusion

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- We introduce the BiSE neuron, that can learn the erosion, dilation and structuring element
- We introduce the BiMoNN, which in theory can learn any composition of dilations and erosions
- We managed to learn perfectly the erosion and dilation
- We managed to learn some structuring elements for opening and closing

Future Work

Future work

- Make the opening and closing work for more structuring elements
- Learn more complicated filters with multiple openings and closings
- Extend the BiMoNN to more complicated operations (for example intersection / union of dilations / erosions)
- Extend the network to classification
- Extend to shape analysis on real data
- Explore different BiSE possibilities

References I

Junior Barrera, Edward R Dougherty, and Nina Sumiko Tomita. Automatic programming of binary morphological machines by design of statistically optimal operators in the context of computational learning theory. *Journal of Electronic Imaging*, 6 (1):54–67, 1997.

Gianni Franchi, Amin Fehri, and Angela Yao. Deep morphological networks. *Pattern Recognition*, 102:107246, 2020.

Alexandre Kirszenberg, Guillaume Tochon, Élodie Puybareau, and Jesus Angulo. Going beyond p-convolutions to learn grayscale morphological operators. In *International Conference on Discrete Geometry and Mathematical Morphology*, pages 470–482. Springer, 2021.

References II

Jonathan Masci, Jesús Angulo, and Jürgen Schmidhuber. A learning framework for morphological operators using counter–harmonic mean. In *International Symposium on Mathematical Morphology and Its Applications to Signal and Image Processing*, pages 329–340. Springer, 2013.

Ranjan Mondal, Deepayan Chakraborty, and Bhabatosh Chanda. Learning 2d morphological network for old document image binarization. In *2019 International Conference on Document Analysis and Recognition (ICDAR)*, pages 65–70. IEEE, 2019.

Ranjan Mondal, Moni Shankar Dey, and Bhabatosh Chanda. Image restoration by learning morphological opening-closing network. *Mathematical Morphology-Theory and Applications*, 4 (1):87–107, 2020.

References III

Makoto Nakashizuka, Shinji Takenaka, and Youji Iiguni. Learning of structuring elements for morphological image model with a sparsity prior. In *2010 IEEE International Conference on Image Processing*, pages 85–88. IEEE, 2010.

Keiller Nogueira, Jocelyn Chanussot, Mauro Dalla Mura, and Jefersson A Dos Santos. An introduction to deep morphological networks. *IEEE Access*, 9:114308–114324, 2021.

Yucong Shen, Xin Zhong, and Frank Y Shih. Deep morphological neural networks. *arXiv preprint arXiv:1909.01532*, 2019.

Stephen S Wilson. Training structuring elements in morphological networks. In *Mathematical Morphology in Image Processing*, pages 1–41. CRC Press, 1993.

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Complementation

If we can force the BiSE neuron to be a dilation, we can learn only the structuring element as well as a complementation.

Definition (Smooth complementation)

Let $\alpha \in [0, 1]$. We define the **smooth complementation** No_α as:

$$No_\alpha : x \in [0, 1] \mapsto \alpha \cdot x + (1 - \alpha) \cdot (1 - x) \tag{13}$$

If we pass a binary image $X \in \{0, 1\}^{\mathbb{Z}^d}$ through the complementation before giving it to a BiSE, then the input is $No_\alpha(X) \in \{\alpha, 1 - \alpha\}^{\mathbb{Z}^d}$.
We need to approximate the bias b to force a dilation.

Dilation Approximation

Proposition

Let $\alpha \in [0,1]$. *Let* $a = \min(\alpha, 1-\alpha)$ *and* $A = \max(\alpha, 1-\alpha)$. $S \subset \mathbb{Z}^d$ *be a structuring element. We consider all the possible images of values in* $\{\alpha, 1-\alpha\}$. *We denote* $\mathcal{V} = \cup_{I \in \{\alpha, 1-alpha\}^{\mathbb{Z}^d}} I \circledast \mathbb{1}_S(\mathbb{Z}^d)$. *Then* $card(\mathcal{V}) = card(S) + 1$. *Let* $v_0 < ... < v_{card(S)} \in \mathcal{V}$. *Then:*

$$\forall i \in \{0, ..., card(S)\}, v_i = a * (card(S) - i) + A * i \tag{14}$$

Definition (Bias dilation function)

The best bias for dilation is $b = \frac{v_0 + v_1}{2}$. Given weights $W \in \mathbb{R}^\Omega$ and a smooth thresholding ξ, we approximate card(S) by $\sum_{i \in \Omega} \xi(W(i))$. We define the **bias dilation function**:

$$b : \alpha \in \mathbb{R} \mapsto \min(\xi(\alpha), 1 - \xi(\alpha))\Big(\sum_{i \in \Omega} W(i) - 1\Big) + 0.5 \tag{15}$$

BiSEC

Definition (BiSEC neuron)

Let $W \in \mathbb{R}^{\Omega}$ be a weight matrix, ξ a smooth thresholding function and $p \in \mathbb{R}_{+}^{*}$ a scaling number. Let $\alpha \in \mathbb{R}$. Let $b(\alpha)$ be the bias dilation function. We define a **BiSEC neuron** as follow, with No_{α} a smooth complementation function:

$$\hat{\epsilon}_{W,\alpha,p} : x \in [0,1]^{\mathbb{Z}^d} \mapsto No_{\xi(\alpha * \infty)} \circ \epsilon_{W,b(\alpha),p} \circ No_{\xi(\alpha)} \tag{16}$$

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Tropical BiSE

We use the fact that the dilation can be written with max instead, inspired from grey-scale morphology.

Proposition (Tropical Dilation)

Let $\Omega \subset \mathbb{Z}^d$ a support kernel and $S \in \Omega$ a structuring element. If $W = -\infty \cdot \mathbb{1}_{\Omega \setminus S}$, then

$$\forall X \subset \mathbb{Z}^d \,,\, \forall j \in \mathbb{Z}^d \,,\, \mathbb{1}_{X \oplus S}(j) = \max_{i \in \Omega}(\mathbb{1}_X(j-i)+W(i)) =: \delta_W(X)_j \quad (17)$$

Definition (Tropical BiSE)

Let $\Omega \subset \mathbb{Z}^d$ be a support kernel and $W \in \mathbb{R}^\Omega$ a set of weights. Let $\alpha \in \mathbb{R}$ and No_α a smooth complementation function. We define the **tropical BiSE** as:

$$\bar{\epsilon}_{W,\alpha} = No_{\xi(\alpha * \infty)} \circ \delta_W \circ No_{\xi(\alpha)} \quad (18)$$

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BiSE Convolution Definition

It is the classic convolution. The difference with BiSE is that we remove the smooth thresholding of weights.

Definition (BiSE Convolution)

Let $\Omega \subset \mathbb{Z}^d$ and $W \in \mathbb{R}^{\Omega}$. Let $b \in \mathbb{R}$ and $p \in \mathbb{R}_+^*$. Let ξ be a smooth thresholding function. We define the **BiSE Convolution** as:

$$CONV_{W,b,p} : I \in [0,1]^{Z^d} = \xi(p(I \circledast W - b)) \tag{19}$$

BiSE Convolution Properties

Proposition (Dilation Equivalence)

We take the same notation as the previous definition. Let $S \subset \Omega$. Then $CONV_{W,b,+\infty}$ is a dilation by S if and only if

$$\max_{K \in \mathcal{P}(\bar{S})} \left(\sum_{i \in K} W_i \right) \leq b < \min_{K \in \mathcal{P}(S), K \neq \emptyset} \left(\sum_{i \in K} W_i \right) \tag{20}$$

Proposition (Erosion Equivalence)

We take the same notation as the previous definition. Let $S \subset \Omega$. Then $CONV_{W,b,+\infty}$ is an erosion by S if and only if

$$\max_{K \in \mathcal{P}(\bar{S})} \left(\sum_{i \in K} W_i \right) + \max_{j \in S} \left(\sum_{i \in S, i \neq j} W_i \right) \leq b < \sum_{i \in S} W_i + \min_{K \in \mathcal{P}(\bar{S})} \left(\sum_{i \in K} W_i \right) \tag{21}$$